Close Tue: 10.2/13.2, 10.3
Close Thu: 13.3 (finish much sooner)
Midterm 1, Thursday, Apr. 20th
Covers 12.1-12.5, 10.1-10.3, 13.1-13.3

Polar

1. Be able to plot points.
2. Be able to convert to Cartesian.
3. Some calculus:

If $r = f(\theta)$, then

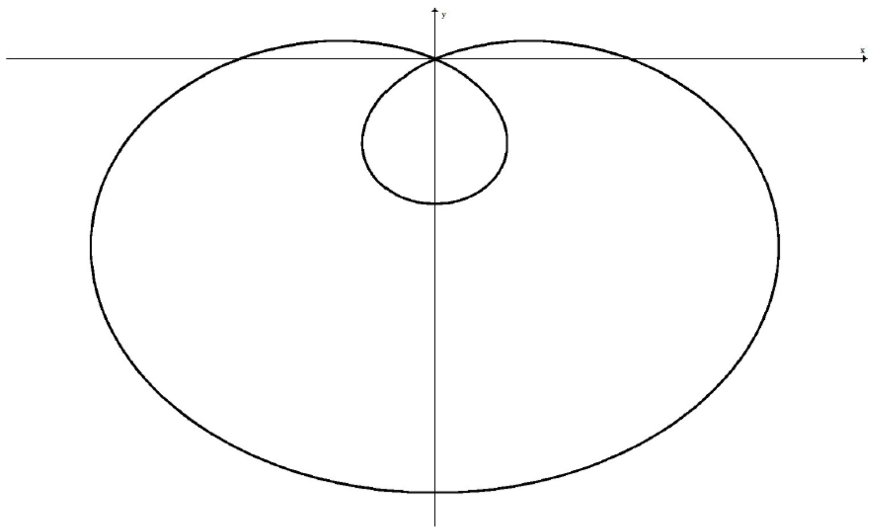
$$x = rcos(\theta) = f(\theta)\cos(\theta)$$
$$y = rsin(\theta) = f(\theta)\sin(\theta)$$

so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$
$$= \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

Example (from an old midterm):
Consider r = 3 – 6sin(θ)

(a) Find (x,y) coordinates of the y-intercepts.
(b) Find the equation for the tangent line at the negative x-intercept. (Put in the form y = mx+b)

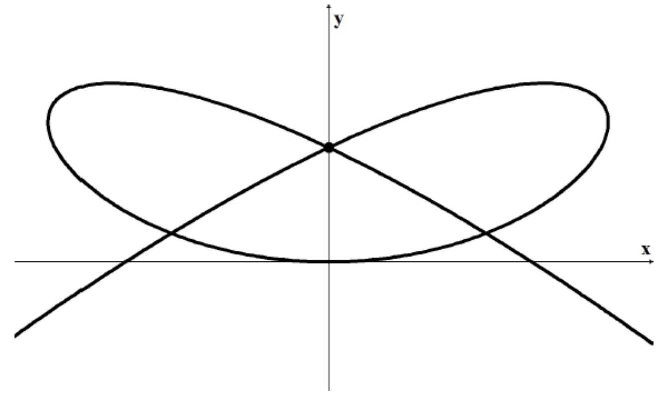

y
x

10.2/13.2, 13.3 Calculus on Curves

Parametric Example (another old exam):
Consider the curve given by

$$x = t^3 - 4t, \; y = 5t^2 - t^4$$

The curve intersects the positive y-axis at the same y-intercept twice. Find the two different tangent slopes at this point.



Distance Traveled on a Curve

The dist. traveled along a curve from $t = a$ to $t = b$ is given by $\int_a^b |\boldsymbol{r}'(t)|dt =$

$$\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\,dt$$

(Note: 2D is same without the z'(t)).

If the curve is "traversed once" we call this **arc length**.

The distance from 0 to *t* is

$$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(u)|du = \text{distance}$$

We call this the **distance/arc length function**.

Note:

$$\frac{ds}{dt} = |\vec{\boldsymbol{r}}'(t)| = \text{speed}$$

Example: x = cos(t), y = sin(t)

(a) Find the distance traveled by this object from t = 0 to t = 6π.

(b) Find the arc length of the path over which this object is traveling.

Example: $x = 3 + 2t$, $y = 4 – 5t$

(a) Find the arc length (from 0 to t).

(b) Reparameterize in terms of $s(t)$.

13.3 (part 1) Curvature

The **curvature** at a point, *K*, is a measure of how quickly a curve is changing direction at that point. That is, we define

$$K = \frac{change\ in\ direction}{change\ in\ arc\ length(distance)}$$

Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

Let

$\overrightarrow{\boldsymbol{T_1}} = unit\ tangent\ vector\ at\ a\ point$

$\overrightarrow{\boldsymbol{T_2}} = unit\ tangent\ vector\ one\ inch\ later$

so

$$K \approx \left|\frac{\overrightarrow{\boldsymbol{T_2}}-\overrightarrow{\boldsymbol{T_1}}}{one\ inch}\right| = \left|\frac{\Delta\vec{\boldsymbol{T}}}{\Delta s}\right|$$

We formally define curvature to be the limit as the distance goes to zero, which gives

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right|$$

Curvature is challenging to compute directly using the definition, so we have some shortcuts.

1st shortcut:

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \left|\frac{d\vec{\boldsymbol{T}}/dt}{ds/dt}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|}$$

2nd shortcut

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^{\mathbf{3}}}$$

And in 2D for a function *y = f(x)*, this becomes

$$K(x) = \frac{|\boldsymbol{f}''(x)|}{\left(\mathbf{1} + \left(\boldsymbol{f}'(\boldsymbol{x})\right)^{\mathbf{2}}\right)^{\mathbf{3/2}}}$$

Proof of short cuts:

Lemma:

$\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{T}}'$ are always orthogonal.

Proof of lemma:

Since $\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}} = \left|\vec{\boldsymbol{T}}\right|^2 = 1$, we can differentiate both sides to get

$$\vec{\boldsymbol{T}}' \cdot \vec{\boldsymbol{T}} + \vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}' = 0.$$

So $2\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}' = 0$. Thus, $\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}' = 0$. (QED)

Theorem: $\frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}'(t)\times\vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3}$

Proof of theorem:

Since $\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|}$, we have

$$\vec{\boldsymbol{r}}'(t) = |\vec{\boldsymbol{r}}'(t)|\vec{\boldsymbol{T}}(t).$$

Differentiating this gives (prod. rule):

$$\vec{\boldsymbol{r}}''(t) = |\vec{\boldsymbol{r}}'(t)|'\vec{\boldsymbol{T}}(t) + |\vec{\boldsymbol{r}}'(t)|\vec{\boldsymbol{T}}'(t).$$

Take cross-prod. of both sides with $\vec{\boldsymbol{T}}$:

$$\vec{\boldsymbol{T}} \times \vec{\boldsymbol{r}}'' = |\vec{\boldsymbol{r}}'|' \left(\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}\right) + |\vec{\boldsymbol{r}}'| \left(\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}'\right).$$

Since $\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}} = < 0, 0, 0 >$ (why?) and $\vec{\boldsymbol{T}} = \frac{\vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}}'|}$, we have

$$\frac{\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|} = |\vec{\boldsymbol{r}}'| \left(\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}'\right).$$

Taking the magnitude gives

$$\frac{|\vec{\boldsymbol{r}}'\times\vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|} = |\vec{\boldsymbol{r}}'| \left|\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}'\right| = |\vec{\boldsymbol{r}}'| \left|\vec{\boldsymbol{T}}\right|\left|\vec{\boldsymbol{T}}'\right| sin\left(\frac{\boldsymbol{\pi}}{\boldsymbol{2}}\right),$$

Since $\left|\vec{\boldsymbol{T}}\right| = 1$, we have

$$\left|\vec{\boldsymbol{T}}'\right| = \frac{\left|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''\right|}{\left|\vec{\boldsymbol{r}}'\right|^2}$$

Therefore

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}.$$

Note:

To find curvature for a 2D function *y = f(x)*, we can form a 3D vector function as follows

$$\vec{\boldsymbol{r}}(x) = \langle x, f(x), 0\rangle$$

so $\vec{\boldsymbol{r}}'(x) = \langle 1, f'(x), 0\rangle$ and

$$\vec{\boldsymbol{r}}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\vec{\boldsymbol{r}}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}'' = \langle 0,0, f''(x)\rangle$$

Thus,

$$K(x) = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}$$

$$= \frac{|\boldsymbol{f}''(x)|}{\left(\boldsymbol{1} + \left(\boldsymbol{f}'(\boldsymbol{x})\right)^{\boldsymbol{2}}\right)^{\boldsymbol{3/2}}}$$
